A Mathematical Model of the Mean-Variance (Risk and Expected Return in Investment)

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Abstract
Modern Portfolio Theory otherwise known as mean-variance theory is a very beneficial concept that has helped a lot of people in attaining their life time dreams. The trend in any investment is to maximize returns and minimize as much as possible risk from the investments. In modelling the optimal portfolio, (6) six investments were chosen comprising Bonds/Gilts, Stocks/Shares/Equities, Commodities and Currencies such as Silver, Crude Oil, UK Gilts, GBP/USD, USD/CHN and Nasdaq Futures. These were daily sales but the data was carefully selected from 1st April, 2006 to 30th September, 2006. The result of the analysis with Excel show that the expected return and sharpe ratio are negative figures while the standard deviation has positive value when all the six stocks were allocated equal value of the investment and no consideration was given to a risk-free investment. It is concluded that in such a case where only a single stock gains the whole allocation of the assets, short-selling would be permitted.

Keywords: *Mean-variance theory, standard deviation, Sharpe ratio, Stocks, Risk.*

Introduction

Several people are allergic to risk and the reason is not far-fetched. The word ‘Risk’ itself knows that it is far from ‘safety’.
The English Oxford Dictionary defines risk as:

> “A situation involving exposure to danger; the possibility that something unpleasant or unwelcome will happen.”

In Economics, Marketing and Management, just to list a few, the term risk is related to variance and standard deviation. That makes it a very useful and important term in determining business prospects financial analysts, entrepreneurs and managers of investments. It is associated. The Mean-Variance Portfolio Theory is a concept that is not different from risk and has been viewed by different scholars and financial analysts alike in different perspective and times but with the same objective. Investopedia.com describes Modern Portfolio Theory as a theory on how risk-averse investors can construct portfolios to optimize expected return based on a given level of market risk, emphasizing that risk is an inherent part of higher reward. Risk and Return are very vital tools if any business venture being it theoretical or actual. In the same vein Hoesli and MacGregor (2002), considers modern portfolio theory as a "Theoretical framework for calculating risk and return when assets are combined in a portfolio." Lin Ju PhD have seen it in the same direction in his handout of Financial Mathematics to master's students on the topic Mean-variance Portfolio Theory and described the context as a method used by investors in choosing efficient portfolio.

Literature Review

One cannot discuss this concept without mention the father of the theory, Harry Markowitz who is known to have discovered and presented it to the world in the 1950s. Investopedia and other well-known academics confirm that the theory was first presented as a paper titled "Portfolio Selection" and printed by the Journal of Finance in 1952. In investment risk and return are basically viewed as sharing certain connections. It is very true that investments with high risks turn out to generate high return and those with low risk turn to harvest low returns as well. Allocating the optimal portfolio seems to date back from the old as a well-established phenomenon which has always attracted the attention of both scholars of academia and entrepreneurs as well as financial experts.

The modern portfolio theory is simply the "trade-off" of the two inseparable concepts; risk and return in an organized format as stressed by (Hoesli and MacGregor, 2000). This report aim at discussing the procedures involved in modelling a good portfolio for a convenient adoption by those that need it in determining which stocks to invest in. They went ahead to buttress their view that in determining and drawing out an efficient portfolio, one has to take into account the very important circumstances as enumerated below:

- **(i)** An excellent portfolio theory must consider valuing the securities that might be included in the portfolio. Special care must be taken in choosing the stocks to be included in the portfolio for a more profitable outcome.
- **(ii)** The calculation of the desired stocks selected for the portfolio.
- **(iii)** The point of calculating systematically to conclusion in other to achieve optimization for the portfolio so that the return will be maximised for a minimised amount of risk.
- **(iv)** Reaching a conclusion by using a suitable financial tool to monitor the portfolio by determining whether it has actually met the expectations and if necessary make amends to individual stocks that mat need to be changed.

It is believed that most investors prefer to accept a multiple asset portfolio than just an asset. The benefits associated with multiple assets portfolio outweigh those of single asset portfolio which is why the Markowitz diversification approaches of 1952 was adopted and is still very much useful. A portfolio is designed for the purpose of maximizing return for a given risk level and minimizing risk for a given level of return (Hoesli and MacGregor, 2000).

Methodology

In modelling the optimal portfolio, (6) six investments were chosen comprising Bonds/Gilts, Stocks/Shares/Equities, Commodities and Currencies such as Silver, Crude Oil, UK Gilts, GBP/USD, USD/CHN and Nasdaq Futures. These were sales daily but the data was carefully selected from 1st April, 2006 to 30th September, 2006. These data were got from Yahoo finance with the analysis focusing on the average of "High" and "Low" daily transactions or sells. The collected data was separated into two equal parts as required. The first half being 30th September, 2016 down to 1st July, 2016 while the second part was from 30th June, 2016 down to 1st April, 2016. Excel was used from the beginning of the data analysis to the finish. The data so collected was analysed by maintaining a 30-day monthly average and 3-month quarterly average. The Markowitz's Mean-Variance of 1952 is still very useful and will continue to remain useful as often time's investors and managers apply it in their businesses. It is obviously no doubt that the optimal portfolio is of immense benefit to the vast number of investors who have used its techniques at one time or the other. That brings to the fore the perception of Jerome et al, (2003) who believes that there have been some recent contributions by some quota advocating for a replacement of the method with "betting technique". His words:

> "An optimal portfolio was derived by Ocone and Keratoza, (1991). this expression involves expectation of random variables depending on the interest rate (IR) and the mark price of risk (MPR) and on unspecified derivatives of variables."

Although modern portfolio theory or mean-variance theory as it is referred to at different time seems to be a reliable concept, others see it basically as a "paper-work" that rarely results in the implementers' desired hopes. This confirms Hoesli and MacGregor's (2002), phrase in their definition of portfolio theory as "Theoretical framework". The market sometimes crashes even with the best efficient portfolios being put in place. Others believe that it is only natural for economic disasters to occur irrespective of how experienced and keen financial experts might have been following the economic trends.

Stephen (2004), view it in his own perspective that even though the asymmetric different pay-offs attract portfolio managers and lure them to indulge in risky projects, they really do not seem to take into account the dependants of utility monotonicity and risk-aversion and classify this as folklore. He backed up his argument that:

> "The common folklore clearly has its genesis that in the observation from option pricing theory that increase in the volatility of an option makes it more valuable (see for example Hangen and Sembet (1981), Smith and Watts (1982), and Smith and Stultz (1985). This is however not the same as making the option more desirable to risk-averse investor" (Stephen, 2004; p828)

It could be seen from history that conscious attempts have to be put in place when preparing the Markowitz Portfolio Theory using the Excel solver especially that the data to be used has to be carefully selected in other to avoid a cumbersome analysis. I personally found it rather difficult at some intervals as the solver would reject to analyse and solve with some constraints on some set of data. This indicates that for some data to be really used at some points at the time of designing a portfolio optimization, data has to be carefully selected to avoid conflict with the already intended result.

The maximum mean frontier is a graphical presentation of the highest mean that could be reached for a given level of variance. Return and risk can be measured on a level of both the individual stock and the whole stock market. So, in drawing out a portfolio of multiple assets, one concentrates on the use of authentic data in generating expected returns rather than past outcomes of the past. It demonstrates how the income returns could be made use of in detecting the market expectations of vital variables in the future.

There is a basic formula for risk and return otherwise referred to as mean-variance of a portfolio of investments and elaborated to cover in the step by step design of the portfolio theory. Investopedia.com simply put the expected portfolio return as the weighted average of the expected returns. The minimum variance frontier is a graph with the least variance that could be targeted and reached for a level of expected return.

In analysing the collected data, the model decided to adopt the Markowitz Portfolio Optimization template of Shane Van Dalsem in full and made use of the formulas. In calculating the expected return for a portfolio the given formula is used:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_p)$$

The variance of the two assets (x and y) portfolio is calculated as:

$$\sigma_p^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y cor(r_x r_y)$$

When the assets to be analysed are more than two, the formula below is then used in solving portfolio to accommodate more of the variables:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j cor(r_i r_j)$$

In calculating the multiple assets portfolio, it becomes very imperative for me to make use of the matrix multiplication to determining the optimal asset weights of the portfolio. This is done by applying the expected return formula for the portfolio which is given below:

$$E(r_p) = w^r R = [w_i \ldots w_j]\begin{bmatrix} E(r_i) \\ \ldots\ldots \\ R(r_j) \end{bmatrix}$$

Here w represents the vector of the weights of all the individual assets from i to j in the portfolio and R is the vector of the expected return of the individual assets from i to j in the portfolio. The formula in Excel is

$$\{= mmult(Transpose(w), R)\}$$

which means that you take the matrix multiplication of the transposed weights to be multiplied by their corresponding individual expected returns. Once the formula is keyed in you do not press "Enter" but simply hold down the "shift" and "Ctrl" keys before pressing the enter "key". This is done when arrays are to be calculated using Excel. The variance of the portfolio is calculated by applying correctly the variables in the following equation:

$\sigma_p^2 = w^T S(w)$. The S here is the variance-covariance matrix while the standard deviation is calculated by using the formula below:

$$\sigma_p = \sqrt{w^T S(w)}$$

which is explained as:

$$\left[[w_i - w_j] \begin{bmatrix} \sigma_{1i} \ldots \sigma_{2j} \\ \ldots\ldots\ldots\ldots \\ \sigma_{1i} \ldots \sigma_{2j} \end{bmatrix} \begin{bmatrix} w \\ \ldots\ldots \\ w_j \end{bmatrix} \right]^{\frac{1}{2}}$$

Here too S is the variance –covariance of matrix of the covariance between each of the assets of the portfolio and is calculated as:

$$\{= sqrt(mmult(mmult(transpose(w), s), w,))\}$$

this is also an array that is keyed into Excel to solve and generate the expected result and so care must be taken as to how the "Enter" key is treated. As it was done while expected return was calculated, the "shift" and "Ctrl" keys must be hold down before the "Enter" key is clicked. That is the correct way to solving arrays on Excel like I stressed earlier on. The variance of an asset's return with the returns for the same asset such as σ_{ij} is the variance of the asset's returns. The optimal weights for assets in the portfolio are the ones that maximize the "sharpe Ratio" value for the portfolio. Dalsam (2016) stressed that the sharpe ratio is calculated as:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

The optimal weight of the assets in the risky portfolio is the mix that generates a portfolio along the efficient frontier that is tangential with the Capital Allocation Line (CAL). This outcome in the CAL with the biggest slope serves as the optimal risky portfolio. He went ahead to reiterate that:

> "The Separation Property states that there are two independent responsibilities involved in portfolio choice property. First you determine the optimal risk portfolio and it is assumed to be the best irrespective of client's risk-aversive. The next thing is the allocation of the capital between the risk-free asset and the risky portfolio which is also based on the client's risk-aversion and all the return rates for the risky portfolio and the risk-free asset.
>
> The CAL is calculated as: $Y^* = \frac{E(r_p) - r_f}{A\sigma^2}$. The Y^* here represents the proportion invested in the portfolio and A serves as a measure of the risk-aversion of the investor."(Dalsam, 2016)

The risk-averse investor would always want to be careful in investing in stocks and which is the good reason why its outcome is determined by subtracting the risk-free asset from the expected return and dividing the result by his/her level of averseness multiplied by the standard deviation and squaring it.

Results:

The result of the analysis with Excel show that the expected return and sharpe ratio are negative figures while the standard deviation has positive value when all the six stocks were allocated equal value of the investment and no consideration was given to a risk-free investment. The second test of analysis of the same set of stocks with the inclusion of the risk-free investment which had been reported to be a portion of money saved with a bank and not a T-Bill, show that the whole investment plan should be allocated to only one stock which is Silver and all the other assets should have zero allocation. At this point the sharpe ratio is maximized and the expected return also becomes positive even though the standard deviation is still relatively higher than the expected return. The attachment below represents the outcome:

Average Quarterly Returns	
Silver	0.016591131
Crude Oil	-0.013316534
UK Guilts	0.02514957
GBP/USD	-0.032261379
USD/CNH	0.001454075
Nasdaq Futures	-0.268573781

Equally-weighted Portfolio	Weight
Silver	0.166
Crude Oil	0.166
UK Gilts	0.166
GBP/USD	0.166
USD/CNH	0.166
Nasdaq Futures	0.17
Sum	1

Sharpe	-1.179056505
Risk-free Rate	0.50%
A	5

Expected Return	-0.046053143
Standard Deviation	0.043299997

Optimal Risky Portfolio	Weight
Silver	1.000000008

Crude Oil	0
UK Gilts	0
GBP/USD	0
USD/CNH	0
Nasdaq Futures	0
Sum	1.000000008

Expected Return	0.016591131
Standard Deviation	0.140917008
Sharpe Ratio	0.082255021

Y*	2.33485E-02

	SILVER	**CRUDE OIL**	**UK GILTS**	**GBP/USD**	**USD/CNH**	**NASDAQ FUTURES**
Average Daily Return	0.000184346	-0.000147961	0.00027944	-0.00035846	1.61564E-05	-0.002984153
Average Monthly Return	0.005530377	-0.004438845	0.00838319	-0.010753793	0.000484692	-0.089524594
Average Quarterly Return	0.016591131	-0.013316534	0.02514957	-0.032261379	0.001454075	-0.268573781
Average Annual Return	0.066364525	-0.053266136	0.100598279	-0.129045517	0.005816301	-1.074295122
Daily Variance	0.000224088	0.000304749	1.6037E-05	2.69605E-05	1.90659E-06	2.23656E-05
Monthly Variance	0.006722626	0.00914246	0.000481109	0.000808814	5.71976E-05	0.000670968
Quarterly Variance	0.020167878	0.027427381	0.001443326	0.002426443	0.000171593	0.002012903
Annual Variance	0.080671512	0.109709524	0.005773305	0.009705771	0.000686371	0.008051612

Variance - Covariance Matrix (Daily)

	Silver	**Crude Oil**	**UK Gilts**	**GBP/USD**	**USD/CNH**	**Nasdaq Futures**
Silver	0.00022064	2.44181E-05	9.48476E-06	1.5101E-05	-5.45954E-06	2.2895E-05
Crude Oil	2.44181E-05	0.00030006	-4.29017E-06	9.80497E-06	1.34536E-06	5.94211E-06
UK Gilts	9.48476E-06	-4.29017E-06	1.57902E-05	-2.04275E-06	-2.45036E-08	8.14371E-06
GBP/USD	1.5101E-05	9.80497E-06	-2.04275E-06	2.65457E-05	-3.0158E-06	2.47372E-06
USD/CNH	-5.45954E-06	1.34536E-06	-2.04275E-06	-3.0158E-06	1.87725E-06	-1.69451E-06

Nasdaq Futures	2.2895E-05	5.94211E-06	8.14371E-06	2.47372E-06	-1.69451E-06	2.20215E-05

Variance - Covariance Matrix (Quartely)

	Silver	**Crude Oil**	**UK Gilts**	**GBP/USD**	**USD/CNH**	**Nasdaq Futures**
Silver	0.019857603	0.002197632	0.000853628	0.001359092	-0.000491359	0.002060547
Crude Oil	0.002197632	0.027005421	-0.000386115	0.000882448	0.000121082	0.000534789
UK Gilts	0.000853628	-0.000386115	0.001421121	-0.000183848	-2.20533E-06	0.000732934
GBP/USD	0.001359092	0.000882448	-0.000183848	0.002389113	-0.000271422	0.000222635
USD/CNH	-0.000491359	0.000121082	-2.20533E-06	-0.000271422	0.000168953	-0.000152506
Nasdaq Futures	0.002060547	0.000534789	0.000732934	0.000222635	-0.000152506	0.001981935

	Min. Variance					Optimum Risky Portfolio			
MEAN	0.045425205	0.047632333	0.442854397	0.442853616	0.442854053	0.104884331	0.110544362	0.119322045	0.127205673
S.D.	0.00820922	0.008250318	0.188703921	0.188703568	0.188703754	0.020219787	0.022026722	0.024967359	0.027713108
SHARPE RATIO (SLOPE)	0.660867375	0.92509562	2.134849107	2.134848966	2.134849175	3.208952287	3.202671819	3.17702991	3.146730162
Silver	0.053675068	0.059736726	0	0	0	0.131526397	0.142977897	0.160744142	0.176700807
Crude Oil	0	0	1.000000859	0.999999012	1	0.064487969	0.070552506	0.079953061	0.08839611
UK Gilts	0	0	0	0	0	0.009357266	0.015518735	0.025070632	0.033649627
GBP/USD	0.081130165	0.078402734	0	0	0	0	0	0	0
USD/CHN	0.842796348	0.842094528	-8.42095E-07	0	0	0.794628372	0.770950862	0.734232166	0.701253456
Nasdaq Futures	0.022398426	0.019766016	0	0	0	0	0	0	0
CAL*	0.290375599	0.273187436	-0.13211279	-0.13211246	-0.132112487	0.095196384	0.087676337	0.076877666	0.067568231

The result for the second three months looks more encouraging compared to those in the first three months though also not quite attractive. It seems the stocks involved in the portfolio were really not on their lucky time so they could not look attractive as they could have been. Nevertheless, the outcome of the assets for the second three months showed a better yield.

It is therefore, recommend that the portfolio for the 2nd three months be chosen for the investor. The outcome is attached below.

Equally-weighted Portfolio	**Weight**
Silver	0.166
Crude Oil	0.166
UK Gilts	0.166
GBP/USD	0.166
USD/CNH	0.166
Nasdaq Futures	0.17
Sum	1

Average Quarterly Returns	
Silver	0.310408234
Crude Oil	0.442854053
UK Guilts	0.074133238
GBP/USD	-0.088380612
USD/CNH	0.043800363
Nasdaq Futures	-0.043772275

		Sharpe	2.263113934
Expected Return	0.122506049		
Standard Deviation	0.054131631	**Risk-free Rate**	0.50%
		A	5

Optimal Risky Portfolio	**Weight**
Silver	0
Crude Oil	1.000000008
UK Gilts	0
GBP/USD	0
USD/CNH	0
Nasdaq Futures	0
Sum	1.000000008

Expected Return	0.442854057
Standard Deviation	0.188703755
Sharpe Ratio	2.346821641
Y*	0.491845025

	Variance - Covariance Matrix (Quartely)					
	Silver	**Crude Oil**	**UK Gilts**	**GBP/USD**	**USD/CNH**	**Nasdaq Futures**
Silver	0.012955084	0.008403381	0.00030848	-0.000541882	-0.000685639	-0.00027481
Crude Oil	0.008403381	0.035609107	-0.001297571	0.00816993	-0.001080534	0.004674476
UK Gilts	0.00030848	0.00030848	0.002185973	-0.001561956	0.00023399	-0.000351672
GBP/USD	-0.000541882	0.00816993	-0.001561956	0.01007593	-0.000965711	0.004148202
USD/CNH	-0.000685639	-0.001080534	0.00023399	-0.000965711	0.000228549	-0.000449997
Nasdaq Futures	-0.00027481	0.004674476	-0.000351672	0.004148202	-0.000449997	0.005574154

Conclusions:

Modern Portfolio Theory otherwise known as mean-variance theory is a very beneficial concept that has helped a lot of people in attaining their life time dreams. It will continue to remain one paramount theory in the field of financial Mathematics. The trend in any investment is to maximize returns and minimize as much as possible risk from the investments. The fact that one stock cannot give the desired output is correct but rather absurd as some portfolios designed generates analysis results containing only one stock. What we would expect is that in such a case where only a single stock gains the whole allocation of the assets, short selling would be permitted.

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